

K-sorted Permutations with Weakly Restricted Displacements

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Abstract. A permutation $\pi = (\pi_1\pi_2\dots\pi_n)$ of $\{1, 2, \dots, n\}$ is called *k-sorted* if and only if $|i - \pi_i| \leq k$, for all $1 \leq i \leq n$. We propose an algorithm for generating the set of all *k-sorted* permutations of $\{1, 2, \dots, n\}$ in *lexicographic* order. An *inversion* occurs between a pair of (π_i, π_j) if $i < j$ but $\pi_i > \pi_j$. Let $I(n, k)$ denote the maximum number of *inversions* in *k-sorted* permutations. For *k-sorted* permutations with *weakly* restricted displacements, i.e., $\lfloor n/2 \rfloor \leq k \leq n-1$, we propose a concise formula of $I(n, k)$ by using the *generating functions* approach.

Key words: *k-sorted* permutation, generating function, inversion, lexicographic order, ordinal representation

1 Introduction

A linear ordering of the elements of the set of n marks $\{1, 2, 3, \dots, n\}$ is called a permutation. Permutations are one of the most important combinatorial objects in computing. Many studies have been done on algorithms for generating permutations [8, 16]. However, little research has been devoted on a special kind of permutations, called *k-sorted* permutations. A *k-sorted* permutation is a class of *restricted* permutation that every mark is located at most k displacements from its *right* place. A *restricted* permutation is a permutation that the positions of the marks are subject to some restraints. The most widely known *restricted* permutations are called *derangements* that are permutations without fixed points, means no mark is located at its *right* place. In general, there are three directions of research in *restricted* permutations: *enumerating*, *generating*, and *analyzing*. *Enumerating* refers to derive a formula that can calculate the number of *restricted* permutations of a class of *restricted* permutations. *Generating* refers to design an algorithm that can generate all *restricted* permutations of a class of *restricted* permutations. *Analyzing* refers to analyze certain property of the *restricted* permutation of a class of *restricted* permutations. Many studies deal with the enumeration of restricted permutations [12, 17, 18, 19]. But, few research on the other two directions. Bongiovanni *et al.* [2] introduced a problem of *quasi sorting* as the one of transferring a given permutation to a *k-sorted* permutation. In that paper, they proposed the bases for the development of algorithms for generating *k-sorted* permutations through element comparisons but did not propose an algorithm for systematically generating all *k-sorted* permutations. Berman [1] and Dutton [4] focus on analyzing *k-sorted* permutations. The motivation of this paper is twofold. First, by using an ordinal representation [9, 10], we propose an algorithm for systematically *generating k-sorted* permutations, in *lexicographic* order. That is, generate all *k-sorted* permutations step-by-step in increasing order. Second, as a measure of disordered, *inversion* of a permutation is a crucial property in the study of permutations [7, 14]. When a list of size n is nearly sorted, a straight insertion sort algorithm is highly efficient since only a number of comparisons equal to the number of inversions in the original list, plus at most $n-1$, is required [4]. So, we *analyze* the maximal number of *inversions* of *k-sorted* permutations. In short, this study focuses on *generating* and *analyzing* a class of *restricted* permutations, called *k-sorted* permutations.

Let S_n denote the set of all permutations of n marks $\{1, 2, \dots, n\}$. That is, a permutation $\pi = (\pi_1\pi_2\dots\pi_n)$ belongs to S_n if and only if

$$\begin{aligned} \pi_i &\in \{1, 2, \dots, n\}, \text{ for all } i = 1, 2, \dots, n, \text{ and} \\ \pi_i &\neq \pi_j, \text{ for all } i \neq j. \end{aligned}$$

For a permutation π , if it satisfies $\pi_i = i$ then we say that the mark i is located at its *right* place. A *k-sorted* permutation is a permutation that satisfies the following definition [4].

Definition 1. A permutation $\pi = (\pi_1\pi_2\dots\pi_n)$ in S_n is called *k-sorted* if and only if

$$|i - \pi_i| \leq k, \text{ for all } 1 \leq i \leq n.$$

In other words, every mark is located at most k displacements from its *right* place.

Let K_n denote the set of all k -sorted permutations of n marks $\{1, 2, \dots, n\}$. It is obvious that $0 \leq k \leq n-1$, and that $K_n \subseteq S_n$. We propose a concise formula of the maximum number of *inversions* in k -sorted permutations with *weakly* restricted displacements. The name of “ k -sorted permutations with *weakly* restricted displacements” was inspired by Lehmer [12]. In that paper, he proposed generating functions for k -sorted permutations that is called permutations with *strongly* restricted displacements, with $k = 1, 2$, and 3 . In this paper, for $\lfloor n/2 \rfloor \leq k \leq n-1$, we call them k -sorted permutations with *weakly* restricted displacements. Notice that n and k are two positive integers throughout this paper. For completeness, let’s include some definitions that are related to it.

Definition 2. In a permutation $\pi = (\pi_1 \pi_2 \dots \pi_n)$, an *inversion* occurs between a pair of (π_i, π_j) if $i < j$ but $\pi_i > \pi_j$.

Definition 3. Let $I(\pi_i)$ denote the number of *inversions* of π_i , then $I(\pi_i)$ is the number of j 's such that $i < j$ but $\pi_i > \pi_j$.

Definition 4. Let $I(\pi)$ denote the number of *inversions* of a permutation $\pi = (\pi_1 \pi_2 \dots \pi_n)$, then

$$I(\pi) = \sum_{i=1}^n I(\pi_i).$$

It is well known that the value of $I(\pi)$ is a measure of disordered in a permutation $\pi = (\pi_1 \pi_2 \dots \pi_n)$. In this study, we focus on the maximum number of *inversions* in K_n .

Definition 5. Let $I(n, k)$ denote the maximum number of *inversions* in K_n , that is

$$I(n, k) = \max \{I(\pi), \text{ for all } \pi \in K_n\}.$$

It is trivial that when $k = 0$, for all n , the number of permutations of K_n is one and $I(n, k)$ is zero. Berman [1] gave a value of $2kn$ as an upper bound of $I(n, k)$. Dutton [4] improved the upper bound down to a value of $0.6kn$ by proposing a formula as follows

$$I(n, k) = 2kn - \min\{f(t_1), f(t_2)\}, \text{ with } t_1 = \left\lfloor \frac{n}{m} \right\rfloor, t_2 = \left\lceil \frac{n}{m} \right\rceil, m = \left\lceil \frac{-1 + \sqrt{8k^2 + 8k + 1}}{2} \right\rceil,$$

and the function $f(t)$ is defined as follows

$$f(t) = t \left[k(k+1) - \left\lfloor \frac{n}{t} \right\rfloor \frac{\left\lfloor \frac{n}{t} \right\rfloor + 1}{2} \right] + n \left\lfloor \frac{n}{t} \right\rfloor. \tag{1}$$

Here, $\lfloor x \rfloor$ (read “the floor of x ”) stands for the greatest integer that less than or equal to x , and $\lceil x \rceil$ (read “the ceiling of x ”) the least integer that greater than or equal to x . Dutton’s solution is a more or less sophisticated approach.

The first contribution of this paper is that we propose an algorithm for generating the set of all k -sorted permutations of n marks $\{1, 2, \dots, n\}$ in *lexicographic* order. The second contribution of this paper is that, by using the *generating functions* approach, for k -sorted permutations of n marks $\{1, 2, \dots, n\}$ with *weakly* restricted displacements, we propose a concise formula of the maximum number of *inversions* in K_n as follows

$$I(n, k) = 2nk - k(k+1) - \frac{n(n-1)}{2}. \tag{2}$$

The rest of this paper is organized as follows. In Section 2, we discuss representation schemes of permutations. In Section 3, we propose an algorithm to generate, in *lexicographic* order, K_n and to compute $I(n, k)$. In Section 4, we present several recurrences of $I(n, k)$'s that are used in the following section. In Section 5, we derive a concise formula of $I(n, k)$ by using the *generating function* approach. Conclusions are summarized in Section 6.

2 Ordinal Representation Scheme

In combinatorics and mathematics, several representation schemes have been used for permutation, such as *two-line* form [6], *cycle* notation [6], *permutation matrix* [3], *inversion vector* [15], *inversion table* [7]. From a

different operational point of view, we proposed a new representation scheme of permutation that is called *ordinal representation* [9, 10]. Now, let's take a briefly look at it.

Definition 6. For a permutation π in the form of *ordinal representation*, that is $[D_n D_{n-1} \cdots D_1]$, it belongs to S_n if and only if

$$1 \leq D_i \leq i, \text{ for all } i = 1, 2, \dots, n.$$

Here, $[D_n D_{n-1} \cdots D_1]$ is called the *ordinal digits* of a permutation π .

The meaning of *ordinal digits* is easy to understand, if we imagine that a permutation is the result of a successive withdrawing of items individually, one after the other without replacement, from an ordered item set of n marks $\{1, 2, \dots, n\}$. At the beginning of withdrawing, there are n choices we can choose to be the first component of π . That is why we have $1 \leq D_n \leq n$. Once we have chosen an item as the first component of π , there are $n-1$ choices left in the ordered item set. So, we have $1 \leq D_{n-1} \leq n-1$. In the end, only one choice left, so we have $1 \leq D_1 \leq 1$. In other words, the component π_{n-i+1} of π is determined by D_i . Furthermore, the value of D_i is one plus the number of items that are less than π_{n-i+1} and to the right of it.

Since each permutation π in S_n corresponds uniquely to an integer q in the range of $[0, n!-1]$, we have the following theorem.

Theorem 1. In S_n , there is a *one-to-one* correspondence between $[D_n D_{n-1} \cdots D_1]$ and $(\pi_1 \pi_2 \dots \pi_n)$.

Proof. Clearly, it is easy to convert an integer q to its *factorial representation* [11]. First, we divide the integer q by $(n-1)!$ and set the quotient to C_{n-1} , then the remainder is divided by $(n-2)!$ and the quotient is set to C_{n-2} , and so on. That is, any integer q between 0 and $n!-1$ can be represented as

$$q = C_{n-1} \times (n-1)! + C_{n-2} \times (n-2)! + \cdots + C_1 \times 1! + C_0 \times 0! . \quad (3)$$

Here, the following constraints

$$0 \leq C_j \leq j, \text{ for all } j = 0, 1, \dots, n-1,$$

are imposed to ensure uniqueness. These C_j 's are called the *factorial digits* of integer q [11]. By *Definition 6* we know that

$$1 \leq D_i \leq i, \text{ for all } i = 1, 2, \dots, n.$$

Hence, we have a *one-to-one* correspondence between D_i and C_j as follows:

$$D_i = C_j + 1, \text{ where } i = j+1,$$

for all $i = 1, 2, \dots, n$. ■

Thus, if we ordering all permutations in S_n in *lexicographic* order, for example when $n = 7$, then we can use the *ordinal digits* [1 1 1 1 1 1 1] to represent the first (i. e., 0th) permutation $\pi = (1 2 3 4 5 6 7)$ and [7 6 5 4 3 2 1] to the last (i. e., 5039th) permutation $\pi = (7 6 5 4 3 2 1)$, respectively. It is easy to see that $\pi_i = D_n$ for all permutations in S_n .

3 Algorithm

Although many algorithms have been done for generating S_n and various permutation problems [8, 16], we know of no published algorithms for generating K_n in *lexicographic* order. By using *ordinal representation*, we have proposed a method for generating S_n in *lexicographic* order [9, 10]. In this study, we extend that method to generate K_n in *lexicographic* order and to compute $I(n, k)$. The computation is based on an interesting property that the number of *inversions* of a permutation π is equal to the summation of π 's *ordinal digits* minus n . This property is demonstrated in the following theorem.

Lemma 1. For a permutation π in the form of *ordinal representation*, $\pi = [D_n D_{n-1} \cdots D_1]$, we have $D_{n-i+1} = I(\pi_i) + 1$, for all $i = 1, 2, \dots, n$.

Proof. By *Definition 3*, we know that $I(\pi_i)$ is the number of j 's such that $i < j$ but $\pi_i > \pi_j$. From the meaning of *ordinal digits* mentioned above, we know that the value of D_i is one plus the number of items that are less than π_{n-i+1} and to the right of it. In other words, we have

$$D_i = I(\pi_{n-i+1}) + 1.$$

That is, $D_{n-i+1} = I(\pi_i) + 1$. ■

Theorem 2. For a permutation π in the form of *ordinal representation*, $\pi = [D_n D_{n-1} \cdots D_1]$, we have $I(\pi) = \sum_1^n D_i - n$.

Proof. By *Definition 4* and *Lemma 1*, we have $I(\pi) = \sum_1^n I(\pi_i) = \sum_1^n D_i - n$. ■

Therefore, by using *ordinal representation* scheme, we can systematically generate the whole K_n in lexicographic order, and by using *Theorem 2*, we can compute $I(\pi)$ directly and immediately. Note that, by *Definitions 3* and *4*, totally $n!$ comparisons are needed to compute $I(\pi)$. These two tasks can be described as the following algorithm.

Algorithm. Generate K_n in lexicographic order and compute $I(n, k)$.

Input: n and k .

Output: K_n and $I(n, k)$.

Begin

$I(n, k) = 0$

For $D_n = 1$ To n

For $D_{n-1} = 1$ To $n - 1$

...

For $D_1 = 1$ to 1

Let item set $A = \{1, 2, \dots, n\}$

For $i = n$ To 1

Retrieve the D_i th item of A

If the D_i th item satisfies *Definition 1* then

Let $\pi_{n-i+1} =$ the D_i th item of A

Delete the D_i th item of A

Goto Next i

Else

Select case i

Case n

Goto Next D_n

Case $n - 1$

Goto Next D_{n-1}

...

Case 1

Goto Next D_1

End Select

Endif

Next i

Output $\pi = (\pi_1 \pi_2 \dots \pi_n)$ and Compute $I(\pi)$

If $I(\pi) > I(n, k)$ Then Let $I(n, k) = I(\pi)$

Next D_1

...

Next D_{n-1}

Next D_n

Output $I(n, k)$

End

By using this algorithm, for $k = 2$ and $3 \leq n \leq 10$, we find the numbers of K_n are 6, 14, 31, 73, 172, 400, 932, and 2177, respectively, and list some of them in Table 1. All of these numbers are same as those numbers described in [18]. In Table 1, we also list their corresponding $I(\pi)$ and $I(n, k)$.

By using this algorithm, for $2 \leq n \leq 39$ and $1 \leq k \leq n - 1$, we find their $I(n, k)$'s. Then according to these $I(n, k)$'s, in Section 4, we present several recurrences of $I(n, k)$'s that are further used for deriving a concise formula of $I(n, k)$ for k -sorted permutations with weakly restricted displacements.

Table 1. K_n for $k = 2$ and $3 \leq n \leq 5$.

	$\pi = (\pi_1 \pi_2 \dots \pi_n)$	π_1	π_2	π_3	π_4	π_5	$I(\pi)$	$I(n, k)$
$n = 3$	1	1	2	3			0	3
	2	1	3	2			1	
	3	2	1	3			1	
	4	2	3	1			2	
	5	3	1	2			2	
	6	3	2	1			3	
$n = 4$	1	1	2	3	4		0	4
	2	1	2	4	3		1	
	3	1	3	2	4		1	
	4	1	3	4	2		2	
	5	1	4	2	3		2	
	6	1	4	3	2		3	
	7	2	1	3	4		1	
	8	2	1	4	3		2	
	9	2	3	1	4		2	
	10	2	4	1	3		3	
	11	3	1	2	4		2	
	12	3	1	4	2		3	
	13	3	2	1	4		3	
	14	3	4	1	2		4	
$n = 5$	1	1	2	3	4	5	0	4
	2	1	2	3	5	4	1	
	3	1	2	4	3	5	1	
	4	1	2	4	5	3	2	
	5	1	2	5	3	4	2	
	6	1	2	5	4	3	3	
	7	1	3	2	4	5	1	
	8	1	3	2	5	4	2	
	9	1	3	4	2	5	2	
	10	1	3	5	2	4	3	
	11	1	4	2	3	5	2	
	12	1	4	2	5	3	3	
	13	1	4	3	2	5	3	
	14	1	4	5	2	3	4	
	15	2	1	3	4	5	1	
	16	2	1	3	5	4	2	
	17	2	1	4	3	5	2	
	18	2	1	4	5	3	3	
	19	2	1	5	3	4	3	
	20	2	1	5	4	3	4	
	21	2	3	1	4	5	2	
	22	2	3	1	5	4	3	
	23	2	4	1	3	5	3	
	24	2	4	1	5	3	4	
	25	3	1	2	4	5	2	
	26	3	1	2	5	4	3	
	27	3	1	4	2	5	3	
	28	3	1	5	2	4	4	
	29	3	2	1	4	5	3	
	30	3	2	1	5	4	4	
	31	3	4	1	2	5	4	

4 Recurrences of $I(n, k)$'s

In this section, we present some examples of $I(n, k)$'s in Table 2, and propose several recurrences that arise naturally from Table 2. By carefully observing the numbers in Table 2, it is not hard to come up with the following recurrences that are corresponding to the diagonals (with gray color) of Table 2. For convenience, with integer $a \geq 1$, we use "the a^{th} diagonal of Table 2" to stand for those numbers that are presented in the a^{th} diagonal of Table 2. For example, the 1th diagonal of Table 2 stands for $\{1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190\}$, and the 2th diagonal of Table 2 stands for $\{1, 4, 8, 13, 19, 26, 34, 43, 53, 64, 76, 89, 103, 118, 134, 151, 169, 188, 208\}$.

In case of the 1th diagonal of Table 2, i.e., $k = n - 1$, we have the following recurrence

$$I(n, n - 1) = I(n - 1, n - 2) + n - 1, \text{ for } n \geq 2, \tag{4}$$

with $I(1, 0) = 0$. In case of the 2th diagonal of Table 2, i.e., $k = n - 2$, we have the following recurrence

$$I(n, n - 2) = I(n - 1, n - 3) + n - 1, \text{ for } n \geq 4, \tag{5}$$

with $I(3, 1) = 1$. In case of the 3th diagonal of Table 2, i.e., $k = n - 3$, we have the following recurrence

$$I(n, n - 3) = I(n - 1, n - 4) + n - 1, \text{ for } n \geq 6, \tag{6}$$

with $I(5, 2) = 4$. Now, it is not difficult to generalize these recurrences as follows.

In case of the a^{th} diagonal of Table 2, i.e., $k = n - a$, we have the following recurrence

$$I(n, n - a) = I(n - 1, n - a - 1) + n - 1, \text{ for } n \geq 2a, \tag{7}$$

with

$$I(2a - 1, a - 1) = (a - 1)^2. \tag{8}$$

Although we have found these recurrences, we are not satisfied yet. Our goal is to find a concise formula of $I(n, k)$ that can give us an answer quickly. This goal is achieved in next section.

Table 2. $I(n, k)$, for $2 \leq n \leq 20$ and $1 \leq k \leq n - 1$.

$n \setminus k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	
2	1																			
3	1	3																		
4	2	4	6																	
5	2	4	8	10																
6	3	6	9	13	15															
7	3	7	9	15	19	21														
8	4	8	12	16	22	26	28													
9	4	9	14	16	24	30	34	36												
10	5	10	16	20	25	33	39	43	45											
11	5	11	17	23	25	35	43	49	53	55										
12	6	12	18	26	30	36	46	54	60	64	66									
13	6	13	20	28	34	36	48	58	66	72	76	78								
14	7	14	22	30	38	42	49	61	71	79	85	89	91							
15	7	15	24	31	41	47	49	63	75	85	93	99	103	105						
16	8	16	25	33	44	52	56	64	78	90	100	108	114	118	120					
17	8	17	26	36	46	56	62	64	80	94	106	116	124	130	134	136				
18	9	18	28	39	48	60	68	72	81	97	111	123	133	141	147	151	153			
19	9	19	30	41	49	63	73	79	81	99	115	129	141	151	159	165	169	171		
20	10	20	32	43	53	66	78	86	90	100	118	134	148	160	170	178	184	188	190	

5 A Formula of $I(n, k)$

In this section, we derive a concise formula of $I(n, k)$ for k -sorted permutations with weakly restricted displacements, i.e., $\lfloor n/2 \rfloor \leq k \leq n - 1$, by using the generating function approach. Usually, a generating function is a power series. The two most common types of generating functions are ordinal generating functions (ogf) and exponential generating functions (egf) [13]. In this paper, we adopt ogf and the result is the following theorem.

Theorem 3. For $\lfloor n/2 \rfloor \leq k \leq n - 1$, with $n \geq 2$, the maximum number of inversions in k -sorted permutations of n marks $\{1, 2, \dots, n\}$ is

$$I(n, k) = 2nk - k(k + 1) - \frac{n(n - 1)}{2}. \tag{9}$$

Proof. It is easy to come up with the formula in case of $k = n - 1$, i.e., the first diagonal of Table 2. By Definition 3, it is reasonable to locate the larger numbers as left as possible and locate the smaller numbers as

right as possible. Since all permutations in S_n are $(n-1)$ -sorted, the maximum number of *inversions* occurs in the last permutation, i.e., $\pi = (n \ n-1 \cdots 2 \ 1)$. Thus,

$$I(n, n-1) = \frac{n(n-1)}{2}. \quad (10)$$

Definitely, the identity (10) is a special case of the identity (9). Now, we start by discussing the second diagonal of Table 2. First, let $A(x)$ denote the *ogf* of the sequence $(a_3, a_4, a_5, \dots, a_i, \dots)$ in the second diagonal of Table 2, i.e., $k = n-2$, as follows

$$A(x) = a_3x^0 + a_4x^1 + a_5x^2 + \cdots + a_ix^{i-3} + \cdots.$$

Then, we can rewrite the corresponding recurrence (5) as

$$a_i = a_{i-1} + i - 1, \text{ for } i \geq 4, \text{ with } a_3 = 1. \quad (11)$$

To find $A(x)$, we multiply both sides of the recurrence (11) by x^i and sum over $i \geq 4$, then we have

$$\sum_{i=4}^{\infty} a_i x^i = \sum_{i=4}^{\infty} a_{i-1} x^i + \sum_{i=4}^{\infty} (i-1) x^i.$$

That is,

$$(A(x) - a_3)x^3 = A(x)x^4 + \frac{x^2}{(1-x)^2} - x^2 - 2x^3.$$

Since $a_3 = 1$, we obtain

$$A(x)(x^3 - x^4) = \frac{x^2}{(1-x)^2} - x^2 - x^3.$$

Thus, we have

$$A(x) = \frac{1+x-x^2}{(1-x)^3}. \quad (12)$$

In order to obtain an explicit formula for the sequence $(a_3, a_4, a_5, \dots, a_i, \dots)$, we have to expand $A(x)$ in a series of partial fraction. Fortunately, it is easy to expand $A(x)$ as follows.

$$\begin{aligned} A(x) &= \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} - \frac{x^2}{(1-x)^3} \\ &= 2 \sum_{i=0}^{\infty} \binom{i+2}{2} x^i - \sum_{i=0}^{\infty} \binom{i+1}{1} x^i - \sum_{i=0}^{\infty} \binom{i}{2} x^i \\ &= \sum_{i=0}^{\infty} \left(\frac{i^2 + 5i + 2}{2} \right) x^i. \end{aligned} \quad (13)$$

Here, the coefficient of x^0 equals to 1, is the value of a_3 , i.e., $I(3, 1)$; the coefficient of x^1 equals to 4, is the value of a_4 , i.e., $I(4, 2)$; the coefficient of x^2 equals to 8, is the value of a_5 , i.e., $I(5, 3)$; and so on.

Similarly, let $B(x)$ denote the *ogf* of the sequence $(b_5, b_6, b_7, \dots, b_i, \dots)$ in the third diagonal of Table 2, i.e., $k = n-3$, as follows

$$B(x) = b_5x^0 + b_6x^1 + b_7x^2 + \cdots + b_ix^{i-5} + \cdots.$$

The corresponding recurrence (6) can be rewritten as

$$b_i = b_{i-1} + i - 1, \text{ for } i \geq 6, \text{ with } b_5 = 4.$$

As before, we have

$$B(x) = \frac{4-3x}{(1-x)^3}. \quad (14)$$

By expanding $B(x)$ in a form of partial fraction, we obtain an explicit formula for the sequence $(b_5, b_6, b_7, \dots, b_i, \dots)$ as

$$B(x) = \sum_{i=0}^{\infty} \left(\frac{i^2 + 9i + 8}{2} \right) x^i. \quad (15)$$

Here, the coefficient of x^0 equals to 4, is the value of b_5 , i.e., $I(5, 2)$; the coefficient of x^1 equals to 9, is the value of b_6 , i.e., $I(6, 3)$; the coefficient of x^2 equals to 15, is the value of b_7 , i.e., $I(7, 4)$, and so on.

Hence, in general, let $F(x)$ denote the *ogf* of the a^{th} diagonal of Table 2, i.e., $k = n-a$, we have

$$F(x) = \sum_{i=0}^{\infty} \left(\frac{i^2 + (4a-3)i + 2(a-1)^2}{2} \right) x^i. \quad (16)$$

Here, the coefficient of x^i is the value of

$$I(2a-1+i, a-1+i).$$

So, we have $n = 2a - 1 + i$, that is $i = n - 2a + 1$. Thus, by replacing a by $n - k$, we have $i = 2k - n + 1$. Finally, by replacing a by $n - k$, and i by $2k - n + 1$, we have

$$I(n, k) = 2nk - k(k+1) - \frac{n(n-1)}{2}. \quad \blacksquare$$

For example, for $n = 9$, what is the maximum number of *inversions* of *6-sorted* permutations? First, since $a = n - k = 3$, we know that $I(9, 6)$ is in the third diagonal of Table 2. Second, since the first number in the third diagonal of Table 2 is the coefficient of x^0 . So, by $i = n - 2a + 1 = 4$, or by $i = k - a + 1 = 4$, we know that $I(9, 6)$ is the fifth number in the third diagonal of Table 2, that is 30.

By (16), the coefficient of x^4 is the value of $I(9, 6)$, we have,

$$I(9, 6) = \frac{4^2 + (4 \times 3 - 3) \times 4 + 2(3-1)^2}{2} = 30.$$

Alternatively, by (9), we also have

$$I(9, 6) = 2 \times 9 \times 6 - 6 \times (6+1) - \frac{9 \times (9-1)}{2} = 30.$$

6 Conclusions

The algorithm we proposed is easy to implement without any preprocessing and aiding by auxiliary data structures. It is quite suit for generating the set of all *k-sorted* permutations of n marks $\{1, 2, \dots, n\}$ in lexicographic order. We derive a concise formula of $I(n, k)$ for permutations with *weakly* restricted displacements, i.e., $\lfloor n/2 \rfloor \leq k \leq n-1$, by using the *generating function* approach. The *generating function* approach is a powerful and elegant way in *analytic combinatorics* [5]. The beauty of the *generating function* approach lies not in the result itself, but rather in its wide applicability. Our results can be extent to develop the formula of the distribution of *inversions* in K_n .

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References

- [1] K. A. Berman, J. L. Paul, *Fundamentals of Sequential and Parallel Algorithms*, PWS Publishing Co., Boston, MA., 1997.
- [2] G. Bongiovanni, F. Luccio, and L. Pagl, "The Problem of Quasi Sorting," *Calcolo*, Vol. 16, No. 4, pp. 415-430., 1979.
- [3] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, Second Edition, The MIT Press, 2001.
- [4] R. D. Dutton, "Inversions in k-sorted Permutations," *Discrete Applied Mathematics*, 87, pp. 49-56., 1998.
- [5] P. Flajolet, and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.
- [6] D. E. Knuth, *The Art of Computer Programming, Volume 1: Fundamental Algorithms*, Second Edition, Addison-Wesley., 1973.
- [7] D. E. Knuth, *The Art of Computer Programming, Volume 3: Sorting and Searching*, Second Edition, Addison-Wesley., 1998.
- [8] D. E. Knuth, *The Art of Computer Programming, Volume 4, Fascicle 2: Generating all Tuples and Permutation*, Addison-Wesley., 2005.

- [9] T. Kuo, "A New Method for Generating Permutations in Lexicographic Order," *Journal of Science and Engineering Technology*, Vol. 5, No. 4, pp. 21-29., 2009.
- [10] T. Kuo, "Using Ordinal Representation for Generating Permutations with a Fixed Number of Inversions in Lexicographic Order," *Journal of Computers*, Vol. 19, No. 4, pp. 1-7., 2009.
- [11] D. H. Lehmer, "The Machine Tools of Combinatorics," *Applied Combinatorial Mathematics*, E. F. Beckenbach, ed., John Wiley & Sons, Inc., N Y, pp.5-31., 1964.
- [12] D. H. Lehmer, "Permutations with Strongly Restricted Displacements," *Combinatorial Theory and Its Applications (Balatonfüred: Hungary)*, II, Colloq. Math. Soc. János Bolyai 4, edited by P. Erdős, A. Rényi and Vera T. Sós, North-Holland, Amsterdam, pp. 755-769.,1969.
- [13] C. L. Liu, *Introduction to Combinatorial Mathematics*, Mcgraw-Hill College, 1968.
- [14] B. H. Margolius, "Permutations with Inversions," *Journal of Integer Sequences*, Vol. 4, Article 01.2.4., 2001.
- [15] E. M. Reingold, J. Nievergelt, and N. Deo, *Combinatorial Algorithms: Theory and Practice*, Prentice-Hall, Inc., 1977.
- [16] R. Sedgewick, "Permutation Generation Methods," *ACM Computing Surveys*, Vol. 9, No. 2, pp.137-163., 1977.
- [17] Kløve, Torleiv. "Generating Functions for the Number of Permutations with Limited Displacement," *The Electronic Journal of Combinatorics*, Vol. 16, No. 1, R104, 2009.
- [18] Baltić, Vladimir. "On the Number of Certain Types of Strongly Restricted Permutations," *Applicable Analysis & Discrete Mathematics*, Vol. 4, No. 1, pp.119-135., 2010.
- [19] Baltić, Vladimir. "Applications of the Finite State Automata for Counting Restricted Permutations and Variations," *The Yugoslav Journal of Operations Research*, Vol. 22, No. 2, pp.183-198., 2012.